A recursive algorithmic construction for spherical codes in dimensions $\mathbb{R}^{2^k}$

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Abstract

We present a new approach to construct spherical codes in dimensions $\mathbb{R}^{2^k}$, based on Hopf foliations. Using the fact that S^{2n-1} spheres are foliated by manifolds $S^{n-1}_{\sin\eta} \times S^{n-1}_{\cos\eta}$ parametrised by $\eta \in]0, \pi/2[$ (Hopf foliation), we distribute points in dimensions $\mathbb{R}^{2^k}$ via a recursive algorithm from a basic construction on $\mathbb{R}^4$. Our procedure outperforms some current constructive methods in several small-distance regimes and constitutes a compromise between optimality and computational effort.

Introduction

A spherical code $\mathcal{C}(M, n)$ is a set of M points on the unit Euclidian sphere in $\mathbb{R}^n$. We address the *sphere packing* problem: given a minimum Euclidean distance $d > 0$, to find the largest possible number M of points on S^{n-1} with all mutual distances at least d. This problem is relevant to digital communication over Gaussian channels and is a generalisation of the phase shift keying (PSK) modulation for dimensions greater than two. Modulations on dimensions 4, 8 and 16 have been studied in the context of optical communications.

Materials and methods

The Hopf fibration is the mapping [2]:

$$\begin{aligned} h : S^{2n-1} &\to S^n \\ (z_0, z_1) &\mapsto (2z_0\overline{z_1}, |z_0|^2 - |z_1|^2) \end{aligned} \tag{1}$$

where z_0, z_1 are elements of a normed division algebra: $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$ ($n = 1, 2, 4, 8$). It gives a foliation of the S^{2n-1} sphere by the product manifolds $S^{n-1}_{\sin\eta} \times S^{n-1}_{\cos\eta}$, themselves products of spheres with respective radii $\sin\eta$ and $\cos\eta$. When $n = 2$, the S^3 sphere is foliated by tori T^2.

$$\begin{aligned} \iota : \left[0, \frac{\pi}{2}\right] \times S^{n-1} \times S^{n-1} &\to S^{2n-1} \\ (\eta; \hat{n}_1, \xi_1; \hat{n}_2, \xi_2) &\mapsto (e^{\hat{n}_1\xi_1} \sin\eta, e^{\hat{n}_2\xi_2} \cos\eta) \end{aligned} \tag{2}$$

The spherical code construction follows a recursive procedure:

1. To vary the parameter η, generating a family of leaves $S^{n-1}_{\sin\eta} \times S^{n-1}_{\cos\eta}$ mutually distant of at least d.
2. On each leaf $S^{n-1}_{\sin\eta} \times S^{n-1}_{\cos\eta}$, to distribute points recursively on each one of the spheres $S^{n-1}_{\sin\eta}$ and $S^{n-1}_{\cos\eta}$, at scaled minimum distance.

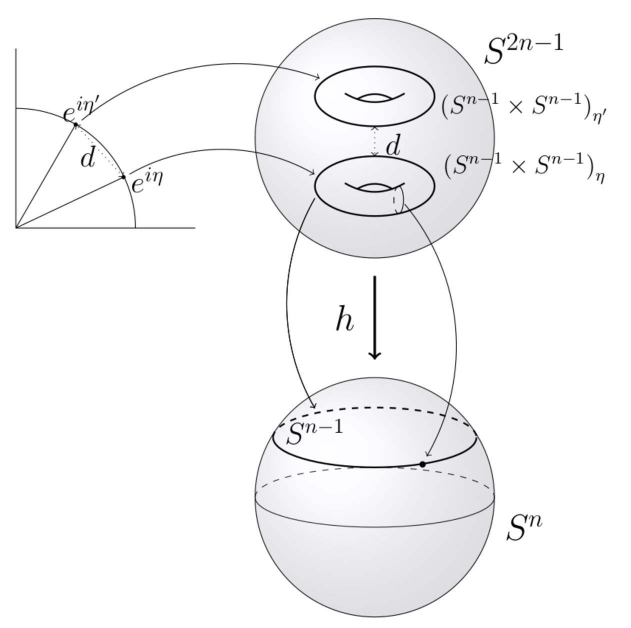

Figure 1: Hopf map and distances between leaves.

When $n = 2$, we choose n internal circles and distribute m points on each of them equidistantly, with an angle of displacement ψ between consecutive internal circles. This distribution is similar to a lattice structure and approaches the hexagonal lattice A_2 when the minimum distance d becomes small.

Results

Using the lattice approximation, it is possible to calculate an upper-bound and asymtotic density for spherical codes by Hopf foliations (SCHF):

$$\Delta_{\text{SCHF}} = [\Delta(\Lambda_2)]^{n/2} \frac{\mathbb{V}_{2n-1}}{(\mathbb{V}_2)^{n/2}(\mathbb{V}_1)^{n-1}}, \tag{3}$$

where $\Delta(\Lambda_2) = 0.9069$ is the density of A_2.

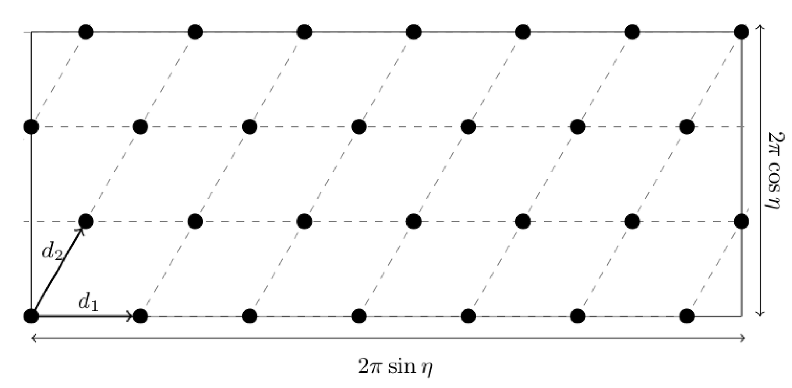

Figure 2: The distribution on each flat torus approaches a lattice when $d \to 0$.

Table 1: SCHF densities, bound and previous dimension lattice bound (Λ-bound).

d	$\mathbb{R}^4$	$\mathbb{R}^8$	$\mathbb{R}^{16}$
0.2	0.565321	0.0333994	7.2458×10^{-6}
0.1	0.584431	0.0428046	1.2878×10^{-5}
0.01	0.602568	0.0492034	2.0036×10^{-5}
SCHF bound	0.6046	0.0492163	2.06946×10^{-5}
Λ-bound	0.74048	0.29530	0.01686

We compare our performance with different implemenations of the TLSC with different subcodes [3] and present the regimes of overperformance.

Table 2: Regimes of SCHF outperformance over TLSC.

n	d	SCHF	TLSC (k)	TLSC (hyperplanes)	TLSC (polygones)
8	0.9	64	8	8	40
	0.8	144	8	8	128
	0.3	104, 512	45, 252	61, 060	89, 945
	0.2	2.28×10^6	3.42×10^5	6.64×10^5	2.15×10^6
16	0.2	6.93×10^{10}	4.76×10^9	7.44×10^9	5.01×10^9
	0.1	4.16×10^{15}	2.41×10^{12}	7.32×10^{12}	2.39×10^{15}
32	0.2	3.40×10^{16}	2.47×10^{16}	3.10×10^{16}	7.06×10^{15}
	0.1	8.66×10^{26}	6.81×10^{21}	1.50×10^{22}	7.02×10^{24}

Conclusions

Our algorithmic spherical codes construction offers a compromise solution between cardinality and computational effort. Although it is not asymptotically dense on higher dimensions, it outperformances some constructive codes in several regimes. The systematic essence of the construction yields advantages for coding and decoding processes.

References

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Acknowledgements

This work was supported by FAPESP grant 16/05126-0.